

On channels with positive quantum zero-error capacity having vanishing n -shot capacity

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Abstract

We show that unbounded number of channel uses may be necessary for perfect transmission of quantum information. For any n we explicitly construct low-dimensional quantum channels (input dimension 4, Choi rank 2 or 4) whose quantum zero-error capacity is positive but the corresponding n -shot capacity is zero. We give estimates for quantum zero-error capacity of such channels (as a function of n) and show that these channels can be chosen in any small vicinity (in the cb -norm) of a classical-quantum channel.

Mathematically, this property means appearance of an ideal (noiseless) subchannel only in sufficiently large tensor power of a channel.

Our approach (using special continuous deformation of a maximal commutative $*$ -subalgebra of M_4) also gives low-dimensional examples of superactivation of 1-shot quantum zero-error capacity.

Finally, we consider multi-dimensional construction which gives channels with greater values of quantum zero-error capacity and vanishing n -shot capacity.

1 Introduction

It is well known that the rate of information transmission over classical and quantum communication channels can be increased by simultaneous use of many copies of a channel. It is this fact that implies necessity of regularization in definitions of different capacities of a channel [7, 12].

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In this paper we show that perfect transmission of quantum information over a quantum channel may require unbounded number of channel uses. We prove by explicit construction that for any given n there is a channel Φ_n such that

$$\bar{Q}_0(\Phi_n^{\otimes n}) = 0, \quad \text{but} \quad Q_0(\Phi_n) > 0, \quad (1)$$

where \bar{Q}_0 and Q_0 are respectively the 1-shot and the asymptotic quantum zero-error capacities defined in Section 2.

This effect is closely related to the recently discovered phenomenon of superactivation of zero-error capacities [1, 4, 5]. Indeed, (1) is equivalent to existence of $m > n$ such that

$$\bar{Q}_0(\Phi_n) = \bar{Q}_0(\Phi_n^{\otimes 2}) = \dots = \bar{Q}_0(\Phi_n^{\otimes(m-1)}) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi_n^{\otimes m}) > 0. \quad (2)$$

Mathematically, (2) means that all the channels $\Phi_n, \Phi_n^{\otimes 2}, \dots, \Phi_n^{\otimes(m-1)}$ have no ideal (noiseless) subchannels but the channel $\Phi_n^{\otimes m}$ has.

We show how for any given n to explicitly construct a pseudo-diagonal quantum channel Φ_n with the input dimension $d_A = 4$ and the Choi rank $d_E \geq 2$ satisfying (2) by determining its noncommutative graph. We also obtain the estimate for m as a function of n , which gives the lower bound for $Q_0(\Phi_n)$ in (1). This shows that

$$\sup_{\Phi} \{ Q_0(\Phi) \mid \bar{Q}_0(\Phi^{\otimes n}) = 0 \} \geq \frac{2 \ln(3/2)}{\pi n} \quad \forall n. \quad (3)$$

It is also observed that a channel Φ_n satisfying (1) and (2) can be obtained by arbitrarily small deformation (in the cb -norm) of a classical-quantum channel with $d_A = d_E = 4$.

The main problem in finding the channel Φ_n is to show nonexistence of error correcting codes for the channel $\Phi_n^{\otimes n}$ (provided the existence of such codes is proved for $\Phi_n^{\otimes m}$). We solve this problem by using the special continuous deformation of a maximal commutative $*$ -subalgebra of 4×4 matrices as the noncommutative graph of Φ_n and by noting that the Knill-Laflamme error-correcting conditions are violated for any maximal commutative $*$ -subalgebra with the positive dimension-independent gap (Lemma 3).

Our construction also gives low-dimensional examples of superactivation of 1-shot quantum zero-error capacity. In particular, it gives an example of symmetric superactivation with $d_A = 4, d_E = 2$ (simplifying the example in [15]) and shows that such superactivation is possible for two channels with

$d_A = d_E = 4$ if one of them is arbitrarily close (in the cb -norm) to a classical-quantum channel.

In the last section we consider multi-dimensional generalization of our basic construction. It gives examples of channels which amplify the lower bound in (3) by the factor $\frac{2\ln 2}{\pi} \approx 2.26$. Unfortunately, we did not managed to show that the value in the left side of (3) is $+\infty$ (as it is reasonable to conjecture). Estimation of this value remains an open question.

2 Preliminaries

Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel, i.e. a completely positive trace-preserving linear map [7, 12]. Stinespring's theorem implies the existence of a Hilbert space \mathcal{H}_E and of an isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \text{Tr}_{\mathcal{H}_E} V \rho V^*, \quad \rho \in \mathfrak{S}(\mathcal{H}_A). \quad (4)$$

The minimal dimension of \mathcal{H}_E is called Choi rank of Φ and denoted d_E .

The quantum channel

$$\mathfrak{S}(\mathcal{H}_A) \ni \rho \mapsto \hat{\Phi}(\rho) = \text{Tr}_{\mathcal{H}_B} V \rho V^* \in \mathfrak{S}(\mathcal{H}_E) \quad (5)$$

is called *complementary* to the channel Φ [7, 8]. The complementary channel is defined uniquely up to isometrical equivalence [8, the Appendix].

The 1-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel Φ is defined as $\sup_{\mathcal{H} \in q_0(\Phi)} \log_2 \dim \mathcal{H}$, where $q_0(\Phi)$ is the set of all subspaces \mathcal{H}_0 of \mathcal{H}_A on which the channel Φ is perfectly reversible (in the sense that there is a channel Θ such that $\Theta(\Phi(\rho)) = \rho$ for all states ρ supported by \mathcal{H}_0). Any subspace $\mathcal{H}_0 \in q_0(\Phi)$ is called *error correcting code* for the channel Φ [6, 7].

The (asymptotic) quantum zero-error capacity is defined by regularization: $Q_0(\Phi) = \sup_n n^{-1} \bar{Q}_0(\Phi^{\otimes n})$ [5, 6].

It is well known that a channel Φ is perfectly reversible on a subspace \mathcal{H}_0 if and only if the restriction of the complementary channel $\hat{\Phi}$ to the subset $\mathfrak{S}(\mathcal{H}_0)$ is completely depolarizing, i.e. $\hat{\Phi}(\rho_1) = \hat{\Phi}(\rho_2)$ for all states ρ_1 and ρ_2 supported by \mathcal{H}_0 [7, Ch.10]. It follows that the 1-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel Φ is completely determined by the set $\mathcal{G}(\Phi) \doteq \hat{\Phi}^*(\mathfrak{B}(\mathcal{H}_E))$ called the *noncommutative graph* of Φ [6]. In particular, the Knill-Laflamme error-correcting condition [9] implies the following lemma.

Lemma 1. A set $\{\varphi_k\}_{k=1}^d$ of unit orthogonal vectors in \mathcal{H}_A is a basis of error-correcting code for a channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ if and only if

$$\langle \varphi_l | A | \varphi_k \rangle = 0 \quad \text{and} \quad \langle \varphi_l | A | \varphi_l \rangle = \langle \varphi_k | A | \varphi_k \rangle \quad \forall A \in \mathfrak{L}, \quad \forall k \neq l, \quad (6)$$

where \mathfrak{L} is any subset of $\mathfrak{B}(\mathcal{H}_A)$ such that $\text{lin} \mathfrak{L} = \mathcal{G}(\Phi)$.

This lemma shows that $\bar{Q}_0(\Phi) \geq \log_2 d$ if and only if there exists a set $\{\varphi_k\}_{k=1}^d$ of unit vectors in \mathcal{H}_A satisfying condition (6).

Remark 1. Since a subspace \mathfrak{L} of the algebra \mathfrak{M}_n of $n \times n$ matrices is a noncommutative graph of a particular channel if and only if

$$\mathfrak{L} \text{ is symmetric } (\mathfrak{L} = \mathfrak{L}^*) \text{ and contains the unit matrix } I_n \quad (7)$$

(see Lemma 2 in [5] or Proposition 2 in [14]), Lemma 1 shows that one can "construct" a channel Φ with $\dim \mathcal{H}_A = n$ having positive (correspondingly, zero) 1-shot quantum zero-error capacity by taking a subspace $\mathfrak{L} \subset \mathfrak{M}_n$ satisfying (7) for which the following condition is valid (correspondingly, not valid)

$$\exists \varphi, \psi \in [\mathbb{C}^n]_1 \text{ s.t. } \langle \psi | A | \varphi \rangle = 0 \text{ and } \langle \varphi | A | \varphi \rangle = \langle \psi | A | \psi \rangle \quad \forall A \in \mathfrak{L}, \quad (8)$$

where $[\mathbb{C}^n]_1$ is the unit sphere of \mathbb{C}^n . \square

We will use the following two notions.

Definition 1. [7] A finite-dimensional¹ channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is called *classical-quantum* if it has the representation

$$\Phi(\rho) = \sum_k \langle k | \rho | k \rangle \sigma_k, \quad (9)$$

where $\{|k\rangle\}$ is an orthonormal basis in \mathcal{H}_A and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$.

Definition 2. [3] A finite-dimensional channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is called *pseudo-diagonal* if it has the representation

$$\Phi(\rho) = \sum_{i,j} c_{ij} \langle \psi_i | \rho | \psi_j \rangle |i\rangle \langle j|,$$

¹In infinite dimensions there exist channels naturally called classical-quantum, which have no representation (9).

where $\{c_{ij}\}$ is a Gram matrix of a collection of unit vectors, $\{|\psi_i\rangle\}$ is a collection of vectors in \mathcal{H}_A such that $\sum_i |\psi_i\rangle\langle\psi_i| = I_{\mathcal{H}_A}$ and $\{|i\rangle\}$ is an orthonormal basis in \mathcal{H}_B .

Pseudo-diagonal channels are complementary to entanglement-breaking channels and vice versa [3, 8].

For any matrix $A \in \mathfrak{M}_n$ denote by Υ_A the operator of Schur multiplication by A in \mathfrak{M}_n (also called the Hadamard multiplication). Its cb -norm will be denoted $\|\Upsilon_A\|_{cb}$. It coincides with the operator norm of Υ_A and is also called the Schur (or Hadamard) multiplier norm of A [11, 13].

3 Basic example

For any given $\theta \in \mathbb{T} \doteq (-\pi, \pi]$ consider the subspace

$$\mathfrak{L}_\theta = \left\{ M = \begin{bmatrix} a & b & \gamma c & d \\ b & a & d & \bar{\gamma} c \\ \bar{\gamma} c & d & a & b \\ d & \gamma c & b & a \end{bmatrix}, a, b, c, d \in \mathbb{C}, \gamma = \exp\left(\frac{i}{2}\theta\right) \right\} \quad (10)$$

of \mathfrak{M}_4 . This subspace satisfies condition (7) and has the following property

$$A = W_4^* A W_4 \quad \forall A \in \mathfrak{L}_\theta, \quad \text{where} \quad W_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (11)$$

Denote by $\widehat{\mathfrak{L}}_\theta$ the set of all channels whose noncommutative graph coincides with \mathfrak{L}_θ . For each θ the set $\widehat{\mathfrak{L}}_\theta$ contains infinitely many different channels with $d_A \doteq \dim \mathcal{H}_A = 4$ and $d_E \geq 2$.

Lemma 2. 1) *There is a family $\{\Phi_\theta^1\}$ of pseudo-diagonal channels (see Def.2) with $d_E = 2$ such that $\Phi_\theta^1 \in \widehat{\mathfrak{L}}_\theta$ for each θ .*

2) *There is a family $\{\Phi_\theta^2\}$ of pseudo-diagonal channels with $d_E = 4$ such that $\Phi_\theta^2 \in \widehat{\mathfrak{L}}_\theta$ for each θ and Φ_0^2 is a classical-quantum channel (see Def.1).*

The families $\{\Phi_\theta^1\}$ and $\{\Phi_\theta^2\}$ can be chosen continuous in the following sense:

$$\Phi_\theta^k(\rho) = \text{Tr}_{\mathcal{H}_E^k} V_\theta^k \rho [V_\theta^k]^*, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \quad k = 1, 2, \quad (12)$$

where V_θ^1, V_θ^2 are continuous families of isometries, $\mathcal{H}_E^1 = \mathbb{C}^2$, $\mathcal{H}_E^2 = \mathbb{C}^4$.²

²This implies continuity of these families in the cb -norm [10].

Lemma 2 is proved in the Appendix by explicit construction of representations (12).

Theorem 1. *Let Φ_θ be a channel in $\widehat{\mathfrak{L}}_\theta$ and $n \in \mathbb{N}$ be arbitrary.*

A) $\bar{Q}_0(\Phi_\theta) > 0$ if and only if $\theta = \pi$ and $\bar{Q}_0(\Phi_\pi) = 1$.

B) If $\theta_1 + \dots + \theta_n = \pi \pmod{2\pi}$ then $\bar{Q}_0(\Phi_{\theta_1} \otimes \dots \otimes \Phi_{\theta_n}) > 0$ and there exist 2^n mutually orthogonal 2-D error correcting codes for the channel $\Phi_{\theta_1} \otimes \dots \otimes \Phi_{\theta_n}$. For each binary n -tuple (x_1, \dots, x_n) the corresponding error correcting code is spanned by the images of the vectors

$$|\varphi\rangle = \frac{1}{\sqrt{2}} [|1\dots 1\rangle + i |2\dots 2\rangle], \quad |\psi\rangle = \frac{1}{\sqrt{2}} [|3\dots 3\rangle + i |4\dots 4\rangle], \quad (13)$$

under the unitary transformation $U_{x_1} \otimes \dots \otimes U_{x_n}$, where $\{|1\rangle, \dots, |4\rangle\}$ is the canonical basis in \mathbb{C}^4 , $U_0 = I_4$ and $U_1 = W_4$ (defined in (11)).

C) If $|\theta_1| + \dots + |\theta_n| \leq 2 \ln(3/2)$ then $\bar{Q}_0(\Phi_{\theta_1} \otimes \dots \otimes \Phi_{\theta_n}) = 0$.

Remark 2. It is easy to show that $\bar{Q}_0(\Phi_\theta^{\otimes n}) = \bar{Q}_0(\Phi_{-\theta}^{\otimes n})$ and that the set of all θ such that $\bar{Q}_0(\Phi_\theta^{\otimes n}) = 0$ is open. Hence for each n there is $\varepsilon_n > 0$ such that $\bar{Q}_0(\Phi_\theta^{\otimes n}) = 0$ if $|\theta| < \varepsilon_n$ and $\bar{Q}_0(\Phi_{\pm\varepsilon_n}^{\otimes n}) > 0$. Theorem 1 shows that $\varepsilon_1 = \pi$ and $2 \ln(3/2)/n < \varepsilon_n \leq \pi/n$ for $n > 1$. Since assertion C is proved by using quite coarse estimates, one can conjecture that $\varepsilon_n = \pi/n$ for $n > 1$. There exist some arguments confirming validity of this conjecture for $n = 2$.

Remark 3. Assertion B of Theorem 1 can be strengthened as follows:

B') If $\theta_1 + \dots + \theta_n = \pi \pmod{2\pi}$ then there exist 2^n mutually orthogonal 2-D projectors $P_{\bar{x}}$ indexed by a binary n -tuple $\bar{x} = (x_1, \dots, x_n)$ such that

$$P_{\bar{x}} A P_{\bar{x}} = \lambda(A) P_{\bar{x}} \quad \forall A \in \mathfrak{L}_{\theta_1} \otimes \dots \otimes \mathfrak{L}_{\theta_n},$$

where $\lambda(A) \in \mathbb{C}$ does not depend on \bar{x} . $P_{\bar{x}}$ is the projector on the subspace $U_{x_1} \otimes \dots \otimes U_{x_n}(\mathcal{H}_0)$, where \mathcal{H}_0 is the linear hull of vectors (13).

So, in the orthonormal basis $\{U_{x_1} \otimes \dots \otimes U_{x_n}|\varphi\rangle, U_{x_1} \otimes \dots \otimes U_{x_n}|\psi\rangle, \dots\}$ the main $2^{n+1} \times 2^{n+1}$ minor of all matrices in $\mathfrak{L}_{\theta_1} \otimes \dots \otimes \mathfrak{L}_{\theta_n}$ has the form

$$\begin{bmatrix} \lambda I_2 & * & \cdots & * \\ * & \lambda I_2 & \cdots & * \\ \cdots & \cdots & \cdots & * \\ * & * & * & \lambda I_2 \end{bmatrix}, \quad \text{where } \lambda \in \mathbb{C}, \quad I_2 \text{ is the unit } 2 \times 2 \text{ matrix.} \quad (14)$$

Theorem 1 implies the main result of this paper.

Corollary 1. *Let n be arbitrary and m be a natural number such that $\theta_* = \pi/m \leq 2\ln(3/2)/n$. Then*

$$\bar{Q}_0(\Phi_{\theta_*}^{\otimes n}) = 0 \quad \text{but} \quad \bar{Q}_0(\Phi_{\theta_*}^{\otimes m}) \geq 1 \quad \text{and hence} \quad Q_0(\Phi_{\theta_*}) \geq 1/m. \quad (15)$$

There exist 2^m mutually orthogonal 2-D error correcting codes for the channel $\Phi_{\theta_}^{\otimes m}$.*

Relation (15) means that it is not possible to transmit any quantum information with no errors by using $\leq n$ copies of the channel Φ_{θ_*} , but such transmission is possible if the number of copies is $\geq m$.

Remark 4. In (15) one can take $\Phi_{\theta_*} = \Phi_{\theta_*}^1$ – a channel from the family described in the first part of Lemma 2. So, Corollary 1 shows that for any n there exists a channel Φ_n with $d_A = 4$ and $d_E = 2$ such that $\bar{Q}_0(\Phi_n^{\otimes n}) = 0$ and

$$Q_0(\Phi_n) \geq \left(\left\lfloor \frac{\pi n}{2\ln(3/2)} \right\rfloor + 1 \right)^{-1} = \frac{2\ln(3/2)}{\pi n} + o(1/n), \quad n \rightarrow +\infty,$$

where $[x]$ is the integer part of x .

It is natural to ask about the maximal value of quantum zero-error capacity of a channel with given input dimension having vanishing n -shot capacity, i.e. about the value

$$S_d(n) \doteq \sup_{\Phi: d_A=d} \{ Q_0(\Phi) \mid \bar{Q}_0(\Phi^{\otimes n}) = 0 \}, \quad (16)$$

where the supremum is over all quantum channels with $d_A \doteq \dim \mathcal{H}_A = d$. We may also consider the value

$$S_*(n) \doteq \sup_d S_d(n) = \lim_{d \rightarrow +\infty} S_d(n) \leq +\infty. \quad (17)$$

The sequences $\{S_d(n)\}_n$ and $\{S_*(n)\}_n$ are non-increasing and the first of them is bounded by $\log_2 d$. Theorem 2 in [15] shows that

$$S_{2d}(1) \geq \frac{\log_2 d}{2} \quad \text{and hence} \quad S_*(1) = +\infty.$$

It seems reasonable to conjecture that $S_*(n) = +\infty$ for all n . A possible way to prove this conjecture is discussed at the end of Section 4.

It follows from the superadditivity of quantum zero-error capacity that

$$S_{d^k}(n) \geq kS_d(nk) \quad \text{and hence} \quad S_*(n) \geq kS_*(nk) \quad \text{for any } k, n. \quad (18)$$

These relations show that the assumption $S_*(n_0) < +\infty$ for some n_0 implies

$$S_d(n) = O(1/n) \quad \text{for each } d \quad \text{and} \quad S_*(n) = O(1/n) \quad \text{if } n \geq n_0.$$

By Corollary 1 we have

$$S_4(n) \geq \left(\left\lceil \frac{\pi n}{2 \ln(3/2)} \right\rceil + 1 \right)^{-1} = \frac{2 \ln(3/2)}{\pi n} + o(1/n), \quad \forall n. \quad (19)$$

This and (18) imply the estimation

$$S_{4^k}(n) \geq k \frac{2 \ln(3/2)}{\pi kn} + o(1/(kn)) = \frac{2 \ln(3/2)}{\pi n} + o(1/(kn)), \quad (20)$$

which shows that

$$S_*(n) \geq \frac{2 \ln(3/2)}{\pi n} \quad \forall n. \quad (21)$$

In Section 4 we will improve these lower bounds by considering the multi-dimensional generalization of the above construction.

Remark 5. Since the parameter θ_* in Corollary 1 can be taken arbitrarily close to zero, the second part of Lemma 2 shows that the channel Φ_{θ_*} , for which $\bar{Q}_0(\Phi_{\theta_*}^{\otimes n}) = 0$ and $Q_0(\Phi_{\theta_*}) > 0$, can be chosen in any small vicinity (in the cb -norm) of the classical-quantum channel Φ_0^2 .

Theorem 1 also gives examples of superactivation of 1-shot quantum zero-error capacity.

Corollary 2. *If $\theta \neq 0, \pi$ then the following superactivation property*

$$\bar{Q}_0(\Phi_\theta) = \bar{Q}_0(\Phi_{\pi-\theta}) = 0 \quad \text{and} \quad \bar{Q}_0(\Phi_\theta \otimes \Phi_{\pi-\theta}) > 0$$

holds for any channels $\Phi_\theta \in \widehat{\mathfrak{L}}_\theta$ and $\Phi_{\pi-\theta} \in \widehat{\mathfrak{L}}_{\pi-\theta}$. For any $\theta \in \mathbb{T}$ there exist 4 mutually orthogonal 2-D error correcting codes for the channel $\Phi_\theta \otimes \Phi_{\pi-\theta}$, one of them is spanned by the vectors

$$|\varphi\rangle = \frac{1}{\sqrt{2}} [|11\rangle + i |22\rangle], \quad |\psi\rangle = \frac{1}{\sqrt{2}} [|33\rangle + i |44\rangle], \quad (22)$$

others are the images of this subspace under the unitary transformations $I_4 \otimes W_4$, $W_4 \otimes I_4$ and $W_4 \otimes W_4$ (the operator W_4 is defined in (11)).

Remark 6. Corollary 2 shows that the channel $\Phi_{\pi/2}^1$ (taken from the first part of Lemma 2) is an example of symmetric superactivation of 1-shot quantum zero-error capacity *with Choi rank 2*.³

By taking the family $\{\Phi_\theta^2\}$ from the second part of Lemma 2 and tending θ to zero we see from Corollary 2 that *the superactivation of 1-shot quantum zero-error capacity may hold for two channels with $d_A = d_E = 4$ if one of them is arbitrarily close (in the cb-norm) to a classical-quantum channel*.

Note that the entangled subspace spanned by the vectors (22) is an error correcting code for the channel $\Phi_0^2 \otimes \Phi_\pi^2$ (and hence for the channel $\Phi_0^2 \otimes \text{Id}_{\mathbb{C}^4}$) despite the fact that Φ_0^2 is a classical-quantum channel.

Proof of Theorem 1. A) It is easy to verify that the subspace \mathfrak{L}_π satisfies condition (8) with the vectors $|\varphi\rangle = [1, i, 0, 0]^\top$, $|\psi\rangle = [0, 0, 1, i]^\top$.

To show that $\bar{Q}_0(\Phi_\theta) = 0$ for all $\theta \neq \pi$ represent the matrix M in (10) as $M = A + cB$, where

$$A = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \bar{\tau} \\ \bar{\tau} & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \end{bmatrix}, \quad \tau = \gamma - 1.$$

$$\text{Let } S = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \text{ then } S^{-1} = S^\top = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$S^{-1}AS = \begin{bmatrix} \tilde{a} & 0 & 0 & 0 \\ 0 & \tilde{b} & 0 & 0 \\ 0 & 0 & \tilde{c} & 0 \\ 0 & 0 & 0 & \tilde{d} \end{bmatrix}, \quad S^{-1}BS = \begin{bmatrix} u & 0 & 0 & v \\ 0 & u & v & 0 \\ 0 & -v & -u & 0 \\ -v & 0 & 0 & -u \end{bmatrix},$$

where

$$\begin{aligned} \tilde{a} &= a - b - c + d, & \tilde{b} &= a + b - c - d, & u &= -\Re\tau = 1 - \Re\gamma \\ \tilde{c} &= a - b + c - d, & \tilde{d} &= a + b + c + d, & v &= i\Im\tau = i\Im\gamma. \end{aligned}$$

³This strengthens the result in [15], where a similar example with Choi rank 3 and the same input dimension was constructed.

Thus the subspace \mathfrak{L}_θ is unitary equivalent to the subspace

$$\mathfrak{L}_\theta^s = \left\{ M = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} + \frac{1}{4}(d + c - b - a) T_\theta, \quad a, b, c, d \in \mathbb{C} \right\} \quad (23)$$

where $T_\theta = S^{-1}BS$ is the above-defined matrix. Hence it suffices to show that condition (8) is not valid for $\mathfrak{L} = \mathfrak{L}_\theta^s$ if $\theta \neq \pi$ (i.e. $\gamma \neq i$).

Assume the existence of unit vectors $|\varphi\rangle = [x_1, x_2, x_3, x_4]^\top$ and $|\psi\rangle = [y_1, y_2, y_3, y_4]^\top$ in \mathbb{C}^4 such that

$$\langle\psi|M|\varphi\rangle = 0 \quad \text{and} \quad \langle\psi|M|\psi\rangle = \langle\varphi|M|\varphi\rangle \quad \text{for all } M \in \mathfrak{L}_\theta^s \quad (24)$$

Since condition (24) is invariant under the rotation

$$|\varphi\rangle \mapsto p|\varphi\rangle - q|\psi\rangle, \quad |\psi\rangle \mapsto \bar{q}|\varphi\rangle + \bar{p}|\psi\rangle, \quad |p|^2 + |q|^2 = 1,$$

we may consider that $y_1 = 0$.

By taking successively $(a = -1, b = c = d = 0)$, $(b = -1, a = c = d = 0)$, $(c = 1, a = b = d = 0)$ and $(d = 1, a = b = c = 0)$ we obtain from (24) the following equations

$$\bar{y}_1 x_1 = \bar{y}_2 x_2 = -\bar{y}_3 x_3 = -\bar{y}_4 x_4 = \frac{1}{4} \langle\psi|T_\theta|\varphi\rangle,$$

$$|x_1|^2 - |y_1|^2 = |x_2|^2 - |y_2|^2 = |y_3|^2 - |x_3|^2 = |y_4|^2 - |x_4|^2 = \frac{1}{4} [\langle\varphi|T_\theta|\varphi\rangle - \langle\psi|T_\theta|\psi\rangle],$$

Since $y_1 = 0$ and $\|\varphi\| = \|\psi\| = 1$, the above equations imply

$$y_1 = y_2 = x_3 = x_4 = 0$$

and

$$|x_1|^2 = |x_2|^2 = |y_3|^2 = |y_4|^2 = \frac{1}{4} [\langle\varphi|T_\theta|\varphi\rangle - \langle\psi|T_\theta|\psi\rangle] = 1/2. \quad (25)$$

So, $|\varphi\rangle = [x_1, x_2, 0, 0]^\top$ and $|\psi\rangle = [0, 0, y_3, y_4]^\top$, where $[x_1, x_2]^\top$ and $[y_3, y_4]^\top$ are unit vectors in \mathbb{C}^2 . It follows from (25) that

$$2 = \left\langle \begin{array}{c} x_1 \\ x_2 \end{array} \middle| \begin{array}{cc} u & 0 \\ 0 & u \end{array} \middle| \begin{array}{c} x_1 \\ x_2 \end{array} \right\rangle - \left\langle \begin{array}{c} y_3 \\ y_4 \end{array} \middle| \begin{array}{cc} -u & 0 \\ 0 & -u \end{array} \middle| \begin{array}{c} y_3 \\ y_4 \end{array} \right\rangle = 2u,$$

which can be valid only if $\gamma = i$, i.e. $\theta = \pi$.

The above arguments also show that $\bar{Q}_0(\Phi_\pi) = 1$, since the assumption $\bar{Q}_0(\Phi_\pi) > 1$ implies, by Lemma 1, existence of orthogonal unit vectors ϕ_1, ϕ_2, ϕ_3 such that condition (24) with $\varphi = \phi_i, \psi = \phi_j$ is valid for all $i \neq j$.

B) Let $M_1 \in \mathfrak{L}_{\theta_1}, \dots, M_n \in \mathfrak{L}_{\theta_n}$ be arbitrary and $X = M_1 \otimes \dots \otimes M_n$. To prove that the linear hull \mathcal{H}_0 of vectors (13) is an error correcting code for the channel $\Phi_{\theta_1} \otimes \dots \otimes \Phi_{\theta_n}$ it suffices, by Lemma 1, to show that

$$\langle \psi | X | \varphi \rangle = 0 \quad \text{and} \quad \langle \psi | X | \psi \rangle = \langle \varphi | X | \varphi \rangle. \quad (26)$$

We have

$$\begin{aligned} 2\langle \psi | X | \varphi \rangle &= \langle 3 \dots 3 | X | 1 \dots 1 \rangle + i\langle 3 \dots 3 | X | 2 \dots 2 \rangle - i\langle 4 \dots 4 | X | 1 \dots 1 \rangle \\ &+ \langle 4 \dots 4 | X | 2 \dots 2 \rangle = c_1 \dots c_n (\bar{\gamma}_1 \dots \bar{\gamma}_n + \gamma_1 \dots \gamma_n) + d_1 \dots d_n (i - i) = 0, \end{aligned}$$

since $\gamma_1 \dots \gamma_n = \pm i$,

$$\begin{aligned} 2\langle \varphi | X | \varphi \rangle &= \langle 1 \dots 1 | X | 1 \dots 1 \rangle + i\langle 1 \dots 1 | X | 2 \dots 2 \rangle - i\langle 2 \dots 2 | X | 1 \dots 1 \rangle \\ &+ \langle 2 \dots 2 | X | 2 \dots 2 \rangle = a_1 \dots a_n (1 + 1) + b_1 \dots b_n (i - i) = 2a_1 \dots a_n \end{aligned}$$

and

$$\begin{aligned} 2\langle \psi | X | \psi \rangle &= \langle 3 \dots 3 | X | 3 \dots 3 \rangle + i\langle 3 \dots 3 | X | 4 \dots 4 \rangle - i\langle 4 \dots 4 | X | 3 \dots 3 \rangle \\ &+ \langle 4 \dots 4 | X | 4 \dots 4 \rangle = a_1 \dots a_n (1 + 1) + b_1 \dots b_n (i - i) = 2a_1 \dots a_n. \end{aligned}$$

Thus the both equalities in (26) are valid.

To prove that the subspace $U_{\bar{x}}(\mathcal{H}_0)$, where $U_{\bar{x}} = U_{x_1} \otimes \dots \otimes U_{x_n}$, is an error correcting code for the channel $\Phi_{\theta_1} \otimes \dots \otimes \Phi_{\theta_n}$ it suffices to note that (11) implies $U_{\bar{x}}^* A U_{\bar{x}} = A$ for all $A \in \mathfrak{L}_{\theta_1} \otimes \dots \otimes \mathfrak{L}_{\theta_n}$.

C) To show that $\bar{Q}_0(\bigotimes_{k=1}^n \Phi_{\theta_k}) = 0$ if $\sum_{k=1}^n |\theta_k| \leq 2 \ln(3/2)$ note that $\mathfrak{L}_\theta = \Upsilon_{D(\theta)}(\mathfrak{L}_0)$ and $\bigotimes_{k=1}^n \mathfrak{L}_{\theta_k} = \bigotimes_{k=1}^n \Upsilon_{D(\theta_k)}(\mathfrak{L}_0^{\otimes n})$, where $\Upsilon_{D(\theta)}$ is the Schur multiplication by the matrix

$$D(\theta) = \begin{bmatrix} 1 & 1 & \gamma & 1 \\ 1 & 1 & 1 & \bar{\gamma} \\ \bar{\gamma} & 1 & 1 & 1 \\ 1 & \gamma & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \bar{\tau} \\ \bar{\tau} & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \end{bmatrix}, \quad (27)$$

where $\tau = \gamma - 1$. By using (27) and Theorem 8.7 in [13] it is easy to show that

$$x_k \doteq \|\Upsilon_{D(\theta_k)} - \text{Id}_4\|_{\text{cb}} \leq |\tau_k| = |1 - \gamma_k| = \left|1 - \exp\left(\frac{i}{2}\theta_k\right)\right| \leq \frac{1}{2}|\theta_k|. \quad (28)$$

Let $\Delta_n \doteq \left\| \bigotimes_{k=1}^n \Upsilon_{D(\theta_k)} - \text{Id}_{4^n} \right\|_{\text{cb}}$. Then by using multiplicativity of the cb -norm and (28) we obtain

$$\Delta_n \leq x_n \prod_{k=1}^{n-1} (1 + x_k) + \Delta_{n-1} \leq \prod_{k=1}^n (1 + x_k) - 1 \leq \prod_{k=1}^n \left(1 + \frac{1}{2}|\theta_k|\right) - 1. \quad (29)$$

Assume that $\bar{Q}_0(\bigotimes_{k=1}^n \Phi_{\theta_k}) > 0$. Then Lemma 1 implies existence of unit vectors φ and ψ in $\mathcal{H}_A^{\otimes n} = \mathbb{C}^{4^n}$ such that

$$\langle \psi | \Psi(A) | \varphi \rangle = 0 \quad \text{and} \quad \langle \varphi | \Psi(A) | \varphi \rangle = \langle \psi | \Psi(A) | \psi \rangle \quad \forall A \in \mathfrak{L}_0^{\otimes n},$$

where $\Psi = \bigotimes_{k=1}^n \Upsilon_{D(\theta_k)}$. Hence for any A in the unit ball of $\mathfrak{L}_0^{\otimes n}$ we have

$$|\langle \psi | A | \varphi \rangle| \leq \Delta_n \quad \text{and} \quad |\langle \varphi | A | \varphi \rangle - \langle \psi | A | \psi \rangle| \leq 2\Delta_n$$

By using (29) and the inequality $x \geq \ln(1+x)$ it is easy to see that the assumption $\sum_{k=1}^n |\theta_k| \leq 2 \ln(3/2)$ implies $\Delta_n \leq 1/2$. So, the above relations can not be valid by the below Lemma 3, since $\mathfrak{L}_0^{\otimes n}$ is a maximal commutative $*$ -subalgebra of \mathfrak{M}_{4^n} . \square

Lemma 3. *Let \mathfrak{A} be a maximal commutative $*$ -subalgebra of \mathfrak{M}_n . Then*

$$\text{either } 2 \sup_{A \in \mathfrak{A}_1} |\langle \psi | A | \varphi \rangle| > 1 \quad \text{or} \quad \sup_{A \in \mathfrak{A}_1} |\langle \varphi | A | \varphi \rangle - \langle \psi | A | \psi \rangle| > 1$$

for any two unit vectors φ and ψ in \mathbb{C}^n , where \mathfrak{A}_1 is the unit ball of \mathfrak{A} .

Proof. Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be the coordinates of φ and ψ in the basis in which the algebra \mathfrak{A} consists of diagonal matrices. Then

$$\sup_{A \in \mathfrak{A}_1} |\langle \psi | A | \varphi \rangle| = \sum_{i=1}^n |x_i| |y_i|, \quad \sup_{A \in \mathfrak{A}_1} |\langle \varphi | A | \varphi \rangle - \langle \psi | A | \psi \rangle| = \sum_{i=1}^n ||x_i|^2 - |y_i|^2|.$$

Let $d_i = |y_i| - |x_i|$. Assume that

$$2 \sum_{i=1}^n |x_i| |y_i| \leq 1 \quad \text{and} \quad \sum_{i=1}^n ||x_i|^2 - |y_i|^2| \leq 1.$$

Since $\sum_{i=1}^n |x_i|^2 = \sum_{i=1}^n |y_i|^2 = 1$, the first of these inequalities implies

$$\left| \sum_{i=1}^n d_i |x_i| \right| \geq 1/2 \quad \text{and} \quad \left| \sum_{i=1}^n d_i |y_i| \right| \geq 1/2.$$

Hence

$$\sum_{i=1}^n \left| |x_i|^2 - |y_i|^2 \right| = \sum_{i=1}^n |d_i| [|x_i| + |y_i|] > \left| \sum_{i=1}^n d_i |x_i| \right| + \left| \sum_{i=1}^n d_i |y_i| \right| \geq 1,$$

where the strict inequality follows from the existence of negative and positive numbers in the set $\{d_i\}_{i=1}^n$. This contradicts to the above assumption. \square

4 Multi-dimensional generalization

Note that

$$\mathfrak{L}_0 = \mathfrak{A}_2^{\otimes 2}, \quad \text{where} \quad \mathfrak{A}_2 = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad a, b \in \mathbb{C} \right\},$$

and that \mathfrak{L}_θ is the image of \mathfrak{L}_0 under the Schur multiplication by matrix (27). So, the above construction can be generalized by considering the corresponding deformation of the maximal commutative $*$ -subalgebra $\mathfrak{L}_0^p = \mathfrak{A}_2^{\otimes p}$ of \mathfrak{M}_{2^p} for $p > 2$. The algebra \mathfrak{L}_0^p can be described recursively as follows:

$$\mathfrak{L}_0^p = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \quad A, B \in \mathfrak{L}_0^{p-1} \right\}, \quad \mathfrak{L}_0^1 = \mathfrak{A}_2.$$

Let $p > 2$ and $\theta \in \mathbb{T} \doteq (-\pi, \pi]$ be arbitrary, $\gamma = \exp(\frac{i}{2}\theta)$. Let $D(\theta)$ be the $2^p \times 2^p$ matrix described as $2^{p-1} \times 2^{p-1}$ matrix $[A_{ij}]$ consisting of the blocks

$$A_{ii} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \forall i, \quad A_{ij} = \begin{bmatrix} \gamma & 1 \\ 1 & \bar{\gamma} \end{bmatrix} \quad \text{if } i < j \quad \text{and} \quad A_{ij} = \begin{bmatrix} \bar{\gamma} & 1 \\ 1 & \gamma \end{bmatrix} \quad \text{if } i > j.$$

Consider the 2^p -D subspace $\mathfrak{L}_\theta^p = \Upsilon_{D(\theta)}(\mathfrak{L}_0^p)$ of \mathfrak{M}_{2^p} ($\Upsilon_{D(\theta)}$ is the Schur multiplication by the matrix $D(\theta)$). This subspace satisfies condition (7) and has the following property

$$A = W_{2^p}^* A W_{2^p} \quad \forall A \in \mathfrak{L}_\theta^p, \tag{30}$$

where W_{2^p} is the $2^p \times 2^p$ matrix having "1" on the main skew-diagonal and "0" on the other places. To prove (30) it suffices to show that it holds for the algebra $\mathfrak{L}_0^p = \mathfrak{A}_2^{\otimes p}$ (by using $W_{2^p} = W_2^{\otimes p}$) and to note that the map $\Upsilon_{D(\theta)}$ commutes with the transformation $A \mapsto W_{2^p}^* A W_{2^p}$.

Denote by $\widehat{\mathfrak{L}}_\theta^p$ the set of all channels whose noncommutative graph coincides with \mathfrak{L}_θ^p . By Proposition 2 in [14] the set $\widehat{\mathfrak{L}}_\theta^p$ contains pseudo-diagonal channels with $d_A = 2^p$ and d_E such that $d_E^2 \geq 2^p$.

Theorem 2. *Let $p > 1$ and $n > 1$ be given natural numbers, Φ_θ be an arbitrary channel in $\widehat{\mathfrak{L}}_\theta^p$ and $\delta_p = \frac{1}{2^{p-1}} \sum_{k=1}^{2^{p-1}} \left| \cot \left(\frac{(2k-1)\pi}{2^p} \right) \right| > 0$.*

A) $\bar{Q}_0(\Phi_\theta^{\otimes n}) = 0$ if $|\theta| \leq \theta_n$, where θ_n is the minimal positive solution of the equation

$$2(1 - \cos(\theta/2)) + \delta_p \sin(\theta/2) = n^{-1} \ln(3/2). \quad (31)$$

B) If $\theta = \pm\pi/n$ then $\bar{Q}_0(\Phi_\theta^{\otimes n}) \geq p - 1$ and there exist 2^n mutually orthogonal 2^{p-1} -D error correcting codes for the channel $\Phi_\theta^{\otimes n}$. For each binary n -tuple (x_1, \dots, x_n) the corresponding error correcting code is spanned by the image of the vectors

$$|\varphi_k\rangle = \frac{1}{\sqrt{2}} [|2k-1 \dots 2k-1\rangle + i |2k \dots 2k\rangle], \quad k = \overline{1, 2^{p-1}}, \quad (32)$$

under the unitary transformation $U_{x_1} \otimes \dots \otimes U_{x_n}$, where $\{|k\rangle\}$ is the canonical basis in \mathbb{C}^{2^p} , $U_0 = I_{2^p}$ and $U_1 = W_{2^p}$ (defined in (30)).

Remark 7. The constant δ_p is the Schur multiplier norm of the skew-symmetric $2^{p-1} \times 2^{p-1}$ matrix having "1" everywhere below the main diagonal. So, the sequence $\{\delta_p\}$ is non-decreasing. It is easy to see that $\delta_2 = 1$, $\delta_3 = \sqrt{2}$, $\delta_4 \approx 1.84$ and that $\delta_p = \left(\frac{2\ln 2}{\pi}\right)p + o(p)$ for large p [11].

Note also that $\theta_n = 2 \ln(3/2) (n\delta_p)^{-1} + o(1/n)$ for large n .

Remark 8. Assertion B of Theorem 2 can be strengthened as follows:

B') If $\theta = \pm\pi/n$ then there exist 2^n mutually orthogonal 2^{p-1} -D projectors $P_{\bar{x}}$ indexed by a binary n -tuple $\bar{x} = (x_1, \dots, x_n)$ such that

$$P_{\bar{x}} A P_{\bar{x}} = \lambda(A) P_{\bar{x}} \quad \forall A \in [\mathfrak{L}_\theta^p]^{\otimes n},$$

where $\lambda(A) \in \mathbb{C}$ does not depend on \bar{x} . $P_{\bar{x}}$ is the projector on the subspace $U_{x_1} \otimes \dots \otimes U_{x_n}(\mathcal{H}_0)$, where \mathcal{H}_0 is the linear hull of vectors (32).

So, in the orthonormal basis $\{U_{x_1} \otimes \dots \otimes U_{x_n} |\varphi_1\rangle, U_{x_1} \otimes \dots \otimes U_{x_n} |\varphi_2\rangle \dots\}$ the main $2^{n+1} \times 2^{n+1}$ minor of all matrices in $[\mathfrak{L}_\theta^p]^{\otimes n}$ has form (14) with I_2 replaced by $I_{2^{p-1}}$.

Proof of Theorem 2. A) Note that $\Upsilon_{D(\theta)} - \text{Id}_{2^p}$ is the Schur multiplication by the matrix

$$-T \otimes \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} + S \otimes \begin{bmatrix} \bar{v} & 0 \\ 0 & v \end{bmatrix},$$

where T is the $2^{p-1} \times 2^{p-1}$ matrix having "0" on the main diagonal and "1" on the other places, S is the $2^{p-1} \times 2^{p-1}$ skew-symmetric matrix having "1" everywhere below the main diagonal, $u = 1 - \Re\gamma = 1 - \cos[\theta/2]$, $v = i\Im\gamma = i\sin[\theta/2]$.

In [11] it is shown that $\|\Upsilon_S\|_{\text{cb}} = 2^{1-p}\|S\|_1 = \delta_p$. Since $\|\Upsilon_T\|_{\text{cb}} \leq 2$ and $\|\Upsilon_{A \otimes B}\|_{\text{cb}} = \|\Upsilon_A \otimes \Upsilon_B\|_{\text{cb}} = \|\Upsilon_A\|_{\text{cb}} \|\Upsilon_B\|_{\text{cb}}$, we have

$$x \doteq \|\Upsilon_{D(\theta)} - \text{Id}_{2^p}\|_{\text{cb}} \leq u\|\Upsilon_T\|_{\text{cb}} + |v|\|\Upsilon_S\|_{\text{cb}} = 2(1 - \cos(\theta/2)) + \delta_p |\sin(\theta/2)|$$

and hence $x \leq n^{-1} \ln(3/2) \leq \sqrt[n]{3/2} - 1$ if $|\theta| \leq \theta_n$.

Assume that $\bar{Q}_0(\Phi_\theta^{\otimes n}) > 0$ for some $\theta \in [-\theta_n, \theta_n]$. By repeating the arguments from the proof of part C of Theorem 1 we obtain

$$|\langle \psi | A | \varphi \rangle| \leq \Delta_n \quad \text{and} \quad |\langle \varphi | A | \varphi \rangle - \langle \psi | A | \psi \rangle| \leq 2\Delta_n \quad (33)$$

for some unit vectors $\varphi, \psi \in \mathbb{C}^{2^{pn}}$ and all A in the unit ball of $[\mathfrak{L}_0^p]^{\otimes n}$, where

$$\Delta_n \doteq \|\Upsilon_{D(\theta)}^{\otimes n} - \text{Id}_{2^{pn}}\|_{\text{cb}} \leq (x + 1)^n - 1 \leq 1/2.$$

Since $[\mathfrak{L}_0^p]^{\otimes n}$ is a maximal commutative $*$ -subalgebra of $\mathfrak{M}_{2^{pn}}$, Lemma 3 shows that (33) can not be valid.

B) Let $\theta = \pm\pi/n$. To prove that the linear hull \mathcal{H}_0 of vectors (32) is an error correcting code for the channel $\Phi_\theta^{\otimes n}$ it suffices, by Lemma 1, to show that

$$\langle \varphi_l | M_1 \otimes \dots \otimes M_n | \varphi_k \rangle = 0 \quad \forall M_1, \dots, M_n \in \mathfrak{L}_\theta^p, \forall k, l$$

and that

$$\langle \varphi_l | M_1 \otimes \dots \otimes M_n | \varphi_l \rangle = \langle \varphi_k | M_1 \otimes \dots \otimes M_n | \varphi_k \rangle \quad \forall M_1, \dots, M_n \in \mathfrak{L}_\theta^p, \forall k, l.$$

Since any matrix in \mathfrak{L}_θ^p can be described as $2^{p-1} \times 2^{p-1}$ matrix $[A_{ij}]$ consisting of the blocks

$$A_{ii} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \forall i \quad \text{and} \quad A_{ij} = \begin{bmatrix} \bar{\gamma}_{ij} c_{ij} & d_{ij} \\ d_{ij} & \gamma_{ij} c_{ij} \end{bmatrix} \quad \forall i \neq j,$$

where $\gamma_{ij} = \exp(is_{ij}\theta/2)$, $s_{ij} = \text{sgn}(j-i)$ and a, b, c_{ij}, d_{ij} are some complex numbers, the above relations are proved by the same way as in the proof of part B of Theorem 1 (by using $\gamma_{ij}^n + \bar{\gamma}_{ij}^n = 0$).

To prove that the subspace $U_{\bar{x}}(\mathcal{H}_0)$, where $U_{\bar{x}} = U_{x_1} \otimes \dots \otimes U_{x_n}$, is an error correcting code for the channel $\Phi_{\theta}^{\otimes n}$ it suffices to note that (30) implies $U_{\bar{x}}^* A U_{\bar{x}} = A$ for all $A \in [\mathfrak{L}_{\theta}^p]^{\otimes n}$. \square

Corollary 3. *Let n be arbitrary and m be a natural number such that $\theta_* = \pi/m \leq \theta_n$. Then*

$$\bar{Q}_0(\Phi_{\theta_*}^{\otimes n}) = 0 \quad \text{but} \quad \bar{Q}_0(\Phi_{\theta_*}^{\otimes m}) \geq p-1 \quad \text{and hence} \quad Q_0(\Phi_{\theta_*}) \geq (p-1)/m.$$

There exist 2^m mutually orthogonal 2^{p-1} -D error correcting codes for the channel $\Phi_{\theta_}^{\otimes m}$.*

Remark 9. Corollary 3 (with Proposition 2 in [14] and Remark 7) shows that for any n there exists a channel Φ_n with $d_A = 2^p$ and arbitrary d_E satisfying the inequality $d_E^2 \geq 2^p$ such that

$$\bar{Q}_0(\Phi_n^{\otimes n}) = 0 \quad \text{and} \quad Q_0(\Phi_n) \geq \frac{p-1}{[\pi/\theta_n] + 1} = \frac{2 \ln(3/2)(p-1)}{\pi n \delta_p} + o(1/n),$$

where $[x]$ is the integer part of x , and hence we have the following lower bounds for the values $S_d(n)$ and $S_*(n)$ (introduced in (16) and (17))

$$S_{2^p} \geq \frac{2 \ln(3/2)(p-1)}{\pi n \delta_p} + o(1/n) \quad \text{and} \quad S_*(n) \geq \frac{2 \ln(3/2)(p-1)}{\pi n \delta_p}$$

(the later inequality is obtained from the former by using relation (18)).

Since $\delta_2 = 1$, the above lower bounds with $p = 2$ coincide with (19)-(21).

Since $\delta_3 = \sqrt{2}$, Remark 9 with $p = 3$ shows that for any n there exists a channel Φ_n with $d_A = 8$ and $d_E = 3$ such that

$$\bar{Q}_0(\Phi_n^{\otimes n}) = 0 \quad \text{and} \quad Q_0(\Phi_n) \geq \sqrt{2} \times \frac{2 \ln(3/2)}{\pi n} + o(1/n).$$

Hence

$$S_8(n) \geq \sqrt{2} \times \frac{2 \ln(3/2)}{\pi n} + o(1/n).$$

Comparing this estimation with (19), we see that the increasing input dimension d_A from 4 to 8 gives the amplification factor $\sqrt{2}$ for the quantum

zero-error capacity of a channel having vanishing n -shot capacity (more precisely, for the lower bound of this capacity).

In general, Remark 9 shows that our construction with the input dimension $d_A = 2^p$ amplifies lower bound (21) for $S_*(n)$ by the factor $\Lambda_p = \frac{p-1}{\delta_p}$. By Remark 7 the non-decreasing sequence Λ_p has a finite limit:

$$\lim_{p \rightarrow +\infty} \Lambda_p = \Lambda_* \doteq \frac{\pi}{2 \ln 2} \approx 2.26.$$

Hence $\Lambda_* \approx 2.26$ is the maximal amplification factor for $S_*(n)$ which can be obtained by increasing input dimension. So, we have

$$S_*(n) \geq \Lambda_* \frac{2 \ln(3/2)}{\pi n} = \frac{\log_2(3/2)}{n} \quad \forall n.$$

Unfortunately, we have not managed to show existence of a channel with *arbitrary* quantum zero-error capacity and vanishing n -shot capacity, i.e. to prove the conjecture $S_*(n) = +\infty$ for all n . This can be explained as follows.

According to Theorem 2, if the input dimension of the channel Φ_θ increases as 2^p then the dimension of error-correcting code for the channel $\Phi_\theta^{\otimes m}$, $\theta = \pi/m$, increases as 2^{p-1} . But simultaneously the norm of the map $\Upsilon_{D(\theta)} - \text{Id}_{2^p}$ characterizing deformation of a maximal commutative $*$ -subalgebra increases as $\delta_p \sin(\theta/2) \sim p\theta/2$ for large p and small θ , so, to guarantee vanishing of the n -shot capacity of Φ_θ by using Lemma 3 we have to decrease the value of θ as $O(1/p)$. Since $\theta = \pi/m$, we see that $\bar{Q}_0(\Phi_\theta^{\otimes m})$ and m have the same increasing rate $O(p)$, which does not allow to obtain large values of $Q_0(\Phi_\theta)$.

Thus, the main obstacle for proving the conjecture $S_*(n) = +\infty$ consists in the unavoidable growth of the norm of the map $\Upsilon_{D(\theta)} - \text{Id}_{2^p}$ as $p \rightarrow +\infty$ (for fixed θ).

First there was a hope to solve this problem by using a freedom in choice of the deformation map $\Upsilon_{D(\theta)}$. Indeed, instead of the matrix $D(\theta)$ introduced before the definition of \mathfrak{L}_θ^p one can use the matrix $D(\theta, S) = [A_{ij}]$ consisting of the blocks

$$A_{ii} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \forall i, \quad A_{ij} = \begin{bmatrix} \gamma & 1 \\ 1 & \bar{\gamma} \end{bmatrix} \quad \text{if } s_{ij} = -1 \quad \text{and} \quad A_{ij} = \begin{bmatrix} \bar{\gamma} & 1 \\ 1 & \gamma \end{bmatrix} \quad \text{if } s_{ij} = 1,$$

where $S = [s_{ij}]$ is *any* skew-symmetric $2^{p-1} \times 2^{p-1}$ matrix such that $s_{ij} = \pm 1$ for all $i \neq j$. For the corresponding subspace $\mathfrak{L}_{\theta, S}^p = \Upsilon_{D(\theta, S)}(\mathfrak{L}_0^p)$ the main

assertions of Theorem 2 are valid (excepting the assertion about 2^m error correcting codes) with the constant δ_p replaced by the norm $\|\Upsilon_S\|_{\text{cb}}$ (in our construction $S = S_*$ is the matrix having "1" everywhere below the main diagonal and $\delta_p = \|\Upsilon_{S_*}\|_{\text{cb}}$). But the further analysis (based on the results from [11]) has shown that

$$\|\Upsilon_S\|_{\text{cb}} \geq \delta_p = \|\Upsilon_{S_*}\|_{\text{cb}}$$

and hence

$$\|\Upsilon_{D(\theta, S)} - \text{Id}_{2^p}\|_{\text{cb}} \geq \|\Upsilon_{D(\theta, S_*)} - \text{Id}_{2^p}\|_{\text{cb}}$$

for any skew-symmetric $2^{p-1} \times 2^{p-1}$ matrix S such that $s_{ij} = \pm 1$ for all $i \neq j$. So, by using the above modification we can not increase the lower bound for $Q_0(\Phi_\theta)$. The useless of some other modifications of the map $\Upsilon_{D(\theta)}$ was also shown.

It is interesting to note that the norm growth of the map $\Upsilon_{D(\theta)} - \text{Id}_{2^p}$ is a *cost of the symmetry requirement* for the subspace \mathfrak{L}_θ^p . Indeed, if we omit this requirement then we would use the matrix $\tilde{D}(\theta) = [A_{ij}]$ consisting of the blocks

$$A_{ii} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \forall i \quad \text{and} \quad A_{ij} = \begin{bmatrix} \gamma & 1 \\ 1 & \bar{\gamma} \end{bmatrix} \quad \forall i \neq j,$$

for which $\|\Upsilon_{\tilde{D}(\theta)} - \text{Id}_{2^p}\|_{\text{cb}} \leq 2|\gamma - 1| \leq \theta$ for all p .

It seems that the above obstacle is technical and can be overcome (within the same construction of a channel) by finding a way to prove the equality $\bar{Q}_0(\Phi^{\otimes n}) = 0$ not using estimations of the distance between the unit balls of $[\mathfrak{L}_\theta^p]^{\otimes n}$ and of $[\mathfrak{L}_\theta^0]^{\otimes n}$. Anyway the question concerning the value

$$S_*(n) \doteq \sup_{\Phi} \{Q_0(\Phi) \mid \bar{Q}_0(\Phi^{\otimes n}) = 0\}$$

remains open.

Appendix: Stinespring representations for the channels Φ_θ^1 and Φ_θ^2

Proof of Lemma 2. Show first that for each θ one can construct basis $\{A_i^\theta\}_{i=1}^4$ of \mathfrak{L}_θ consisting of positive operators with $\sum_{i=1}^4 A_i^\theta = I_4$ such that:

- 1) the function $\theta \mapsto A_i^\theta$ is continuous for $i = \overline{1, 4}$;

2) $\{A_i^\theta\}_{i=1}^4$ consists of mutually orthogonal 1-rank projectors.

Note that \mathfrak{L}_θ is unitary equivalent to the subspace \mathfrak{L}_θ^s defined by (23).

Denote by $\|T_\theta\|$ the operator norm of the matrix T_θ involved in (23). Note that the function $\theta \mapsto T_\theta$ is continuous, $T_0 = 0$ and $\|T_\theta\| \leq \|T_\pi\| = 2$. Let

$$\begin{aligned}\tilde{A}_1^\theta &= \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} - \frac{1}{4}(\alpha - \beta) T_\theta, & \tilde{A}_2^\theta &= \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} - \frac{1}{4}(\alpha - \beta) T_\theta, \\ \tilde{A}_3^\theta &= \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} + \frac{1}{4}(\alpha - \beta) T_\theta, & \tilde{A}_4^\theta &= \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} + \frac{1}{4}(\alpha - \beta) T_\theta\end{aligned}$$

be operators in \mathfrak{L}_θ^s , where $\beta = \min\{\frac{3}{16}, \frac{1}{4}\|T_\theta\|\}$ and $\alpha = 1 - 3\beta$. It is easy to verify that $\tilde{A}_i^\theta \geq 0$ for all i and $\sum_{i=1}^4 \tilde{A}_i^\theta = I_4$. Then $\{A_i^\theta = S\tilde{A}_i^\theta S^{-1}\}_{i=1}^4$, where S is the unitary matrix defined before (23), is a required basis of \mathfrak{L}_θ .

Let $m \geq 2$ and $\{|\psi_i\rangle\}_{i=1}^4$ be a collection of unit vectors in \mathbb{C}^m such that $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^4$ is a linearly independent subset of \mathfrak{M}_m . It is easy to show (see the proof of Corollary 1 in [14]) that \mathfrak{L}_θ is a noncommutative graph of the pseudo-diagonal channel

$$\Phi_\theta(\rho) = \text{Tr}_{\mathbb{C}^m} V_\theta \rho V_\theta^*,$$

where

$$V_\theta : |\varphi\rangle \mapsto \sum_{i=1}^4 [A_i^\theta]^{1/2} |\varphi\rangle \otimes |i\rangle \otimes |\psi_i\rangle$$

is an isometry from $\mathcal{H}_A = \mathbb{C}^4$ into $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^m$ ($\{|i\rangle\}$ is the canonical basis in \mathbb{C}^4). By property 1 of the basis $\{A_i^\theta\}_{i=1}^4$ the function $\theta \mapsto V_\theta$ is continuous.

The first part of Lemma 2 follows from this construction with $m = 2$.

To prove the second part assume that $m = 4$ and $|\psi_i\rangle = |i\rangle$, $i = \overline{1, 4}$. Property 2 of the basis $\{A_i^\theta\}_{i=1}^4$ implies

$$V_0 |\varphi\rangle = \sum_{i=1}^4 \langle e_i | \varphi \rangle |e_i\rangle \otimes |i\rangle \otimes |i\rangle,$$

where $\{|e_i\rangle\}_{i=1}^4$ is an orthonormal basis in \mathbb{C}^4 . Hence $\Phi_0(\rho) = \sum_{i=1}^4 \langle e_i | \rho | e_i \rangle \sigma_i$, $\sigma_i = |e_i \otimes i\rangle \langle e_i \otimes i|$, is a classical-quantum channel.

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Note Added: After publication of the first version of this paper the analogous result concerning quantum ε -error capacity has been appeared [2].

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